Lecture 9 **2012/11/26**

Outline

1. **Motivation**
2. **Local Stability by Linearization**
3. **Examples for Stability by Linearization**
4. **Summary**

1. Motivation

- Stability by linearization is a useful method to study stability of autonomous (dynamic) systems only near an isolate equilibrium. It is a local result.

2. Local Stability by Linearization

1) Linearization

Consider the autonomous (dynamic) system given by

$$x' = f(x),$$

where $x \in B_r(0) = \{x \in R^n \mid \|x\| \le r\}$ and $f \in C^1(B_r(0))$ with $f(0) = 0$, i.e. $x = 0$ is an equilibriumt; or $x(t) = 0\ (t \ge 0)$ is a constant solution of $x' = f(x)$.

Since $f \in C^1(B_r(0))$, using Tailor expansion, we know that

$$f(x) = f(0) + Df(0)x + g(x) = Df(0)x + g(x),$$

where $Df(0) = \frac{\partial f}{\partial x}|_{x=0}$ and $g(x) = f(x) - Df(0)x$, satisfying $g(0) = 0$;

$$g(x) = g(x) - g(0) = \int_0^1 \frac{d}{ds} g(sx)ds = \int_0^1 \{\frac{d}{ds} f(sx) - Df(0)x\}ds$$

$$= \int_0^1 \{\frac{d}{d(sx)} f(sx)x - Df(0)x\}ds = \int_0^1 \{Df(sx)x - Df(0)x\}ds$$

$$= \int_0^1 \{Df(sx) - Df(0)\}x\,ds.$$

It is noted that $\|sx\| \le \|x\|$ as $0 \le s \le 1$. Thus

$$\|g(x)\| = \sup_{\{y: \|y\| \le \|x\|\}} \|Df(y) - Df(0)\| \|x\|.$$

Then

$$\lim_{\|x\|\to 0}\frac{\|g(x)\|}{\|x\|}=\lim_{\|x\|\to 0}\sup_{\{y:\|y\|\le\|x\|\}}\|Df(y)-Df(0)\|=0.$$

Therefore its linearized system is given by

$$x' = Ax,$$

where $A = Df(0)$.

2) Linear Systems with Perturbation

Consider

$$x' = Ax + g(t, x),$$

where $g(t, x)$ is continuous and locally Lipschitz in U containing the origin.

Theorem 9.1 (Stability Theorem) Let $g(t, x)$ be continuous and locally Lipschitz in U containing the origin. If

$$\lim_{\|x\|\to 0}\frac{\|g(t, x)\|}{\|x\|}=0,$$

holds uniformly in t, where $t \ge t_0 \ge 0$ and A has all eigenvalues with negative real part, i.e. $\operatorname{Re}\lambda_j(A) < 0$ for $j = 1, 2, \cdots, n$, then $x = 0$ of $x' = Ax + g(t, x)$ is uniformly asymptotically stable.

Proof. Since $\lim_{\|x\|\to 0}\frac{\|g(t, x)\|}{\|x\|}=0$ holds uniformly, there exists $\varepsilon > 0$ for any given $b > 0$ such that

$$\|g(t, x)\| \le b\|x\| \quad \text{for all} \quad t \ge 0,$$

provided $\|x\| \le \varepsilon$. Then $g(t, 0) \equiv 0$ for all $t \ge 0$, i.e. $x = 0$ is equilibrium of $x' = Ax + g(t, x)$. Since $\operatorname{Re}\lambda_j(A) < 0$ for $j = 1, 2, \cdots, n$, we can find $K > 0$ and $\mu > 0$ (Clue: using the formula of e^{At}) s.t.

$$\|e^{A(t-t_0)}\| \le K\, e^{-\mu(t-t_0)} \quad \text{for} \quad t \ge t_0.$$

For any $\|x_0\| \le \delta = \frac{\varepsilon}{K}$ and $t_0 \ge 0$, there exists a unique solution of

$x' = Ax + g(t, x)$, denoted by $x(t, t_0, x_0)$ for $t \in [t_0, \omega_+)$. We will show that $\omega_+ = \infty$. We apply the formula given by Problem 2 of Homework in Lecture 6 to get

$$x(t,t_0, x_0) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}g(s, x(s,t_0, x_0))ds, \quad t \in [t_0, \omega_+).$$

Then,

$$\| x(t,t_0, x_0)\| \le Ke^{-\mu(t-t_0)} \| x_0 \| + \int_{t_0}^{t} Ke^{-\mu(t-s)} \| g(s, x(s,t_0, x_0))\| ds, \quad t \in [t_0, \omega_+).$$

As long as $\| x(t, t_0, x_0)\| \le \varepsilon$ for all $t \in [t_0, \omega_+)$, then

$$\| g(t, x(t, t_0, x_0))\| \le b \| x(t, t_0, x_0)\|, \quad t \in [t_0, \omega_+)$$

So we have

$$\| x(t,t_0, x_0)\| \le Ke^{-\mu(t-t_0)} \| x_0 \| + Kb\int_{t_0}^{t} e^{-\mu(t-s)} \| x(s,t_0, x_0)\| ds, \quad t \in [t_0, \omega_+).$$

Multiplying $e^{\mu(t-t_0)}$ on both sides, we have

$$e^{\mu(t-t_0)} \| x(t,t_0, x_0)\| \le K \| x_0 \| + Kb\int_{t_0}^{t} e^{\mu(s-t_0)} \| x(s,t_0, x_0)\| ds, \quad t \in [t_0, \omega_+).$$

By Gronwall inequality we obtain

$$e^{\mu(t-t_0)} \| x(t,t_0, x_0)\| \le K \| x_0 \| e^{Kb(t-t_0)}, \quad t \in [t_0, \omega_+),$$

i.e.

$$\| x(t,t_0, x_0)\| \le K \| x_0 \| e^{-(\mu-Kb)(t-t_0)}, \quad t \in [t_0, \omega_+).$$

Since $b > 0$ is arbitrarily and only local result we concern, we choose $b = \dfrac{\mu}{2K} > 0$, and then we have

$$\| x(t,t_0, x_0)\| \le K \| x_0 \| e^{-\frac{\mu}{2}(t-t_0)} \le \varepsilon e^{-\frac{\mu}{2}(t-t_0)} < \varepsilon, \quad t \in [t_0, \omega_+).$$

It follows that $\omega_+ = \infty$ by Extension Theorem. Since $\delta > 0$ and $b > 0$ are independent of $t_0 \ge 0$, $x = 0$ is uniformly asymptotically stable, in fact it is exponentially stable. □

Theorem 9.2 (Unstability Theorem) Let $g(t, x)$ be continuous and locally Lipschitz in U containing the origin. If

$$\lim_{\|x\| \to 0} \frac{\| g(t, x)\|}{\| x \|} = 0,$$

holds uniformly in t, where $t \ge t_0 \ge 0$ and A has at least one eigenvalue with positive real part, i.e. $\operatorname{Re}\lambda_{j_0}(A) > 0$, then $x = 0$ of $x' = Ax + g(t, x)$ is unstable.

Proof. Suppose that A has eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ repeated according to their (algebraic) multiplicity. There exists an invertible matrix P such that $A = PJP^{-1}$, where J is in Jordan normal form, i.e. $J = diag(J_i)$, where

$$J_i = \begin{pmatrix} \lambda_{n_{i-1}+1} & 1 & 0 & \cdots & 0 \\ 0 & \lambda_{n_{i-1}+2} & 1 & \vdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \lambda_{n_i-1} & 1 \\ 0 & \cdots & 0 & 0 & \lambda_{n_i} \end{pmatrix}_{n_i \times n_i},$$

where $i = 1, 2, \cdots, r$; $\sum_{i=1}^{r} n_i = n$; $n_0 = 0$. In order to get the elements of the off diagonal line of J_i, all are the given scalar $\eta > 0$, we need a technique to introduce a matrix given by

$$H = diag(\eta, \eta^2, \cdots, \eta^n),$$

which has $H^{-1} = diag(\eta^{-1}, \eta^{-2}, \cdots, \eta^{-n})$. Then,

$$C = H^{-1}JH = diag(\overline{J}_i),$$

where

$$\overline{J}_i = \begin{pmatrix} \lambda_{n_{i-1}+1} & \eta & 0 & \cdots & 0 \\ 0 & \lambda_{n_{i-1}+2} & \eta & \vdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \lambda_{n_i-1} & \eta \\ 0 & \cdots & 0 & 0 & \lambda_{n_i} \end{pmatrix}_{n_i \times n_i}.$$

Then, by the transformation $x = PHy$, $x' = Ax + g(t, x)$ is transformed into the form of

$$y' = Cy + h(t, y),$$

where $h(t, y) = H^{-1}P^{-1}g(t, PHy)$. Since $\| g(t, x) \| \le b \| x \|$ for $\| x \| \le \delta$, it follows that

$$\| h(t,y) \| \le \| H^{-1}P^{-1} \| \| PH \| b \| y \| \text{ for } \| y \| \le \frac{\delta}{\| PH \|}.$$

We now need only to prove that $y(t)$ will leaves away from the origin when $y(t)$ is in the neighborhood of the origin.

The i^{th} component of $y' = Cy + h(t,y)$ has the form of either

$$y_i' = \lambda_i y_i + h_i(t,y)$$

or

$$y_i' = \lambda_i y_i + \eta y_{i+1} + h_i(t,y).$$

Denoted by j the indices for which $\operatorname{Re}\lambda_j > 0$, by k the indices for which $\operatorname{Re}\lambda_k \le 0$. Let $R(t) = \sum_j |y_j(t)|^2$, $r(t) = \sum_k |y_k(t)|^2$ and choose $\eta > 0$ s.t.

$$0 < 6\eta < \operatorname{Re}\lambda_j \text{ for all } j;$$

and then for the given $\eta > 0$, there exists $0 < \delta << 1$ s.t.

$$\| h(t,y) \| \le \eta \| y \| \text{ for } \| y \| \le \delta.$$

Suppose that $y(t)$ is a solution of $y' = Cy + h(t,y)$ with $y(t_0) = y_0$ satisfying $\| y_0 \| \le \delta$ and $r(t_0) < R(t_0)$. Then, as long as $\| y(t) \| \le \delta$ and $r(t) < R(t)$, notice that $y_j(t)$ may be complex valued because of a possible complex value of λ_j, we have

$$\begin{aligned} R'(t) &= \{\sum_j |y_j(t)|^2\}' = \sum_j \{(y_j(t)\cdot \bar{y}_j(t))^2\}' = 2\sum_j (y_j'(t)\cdot \bar{y}_j(t) + y_j(t)\bar{y}_j'(t)) \\ &= 2\sum_j \operatorname{Re}(y_j'(t)\cdot \bar{y}_j(t)) \\ &= 2\sum_j \{\operatorname{Re}(\lambda_j y_j(t)\cdot \bar{y}_j(t)) + [\eta \operatorname{Re}(y_{j+1}(t)\cdot \bar{y}_j(t))] + \operatorname{Re}(\bar{y}_j(t) h_j(t,y))\}, \end{aligned}$$

where the term in brackets [ ] appears or not appears. By Cauchy-Schwartz inequality we have

$$|\sum_j \operatorname{Re}(y_{j+1}(t)\cdot \bar{y}_j(t))| \le \sum_j |y_{j+1}(t)\cdot \bar{y}_j(t)| \le \sqrt{\sum_j |y_{j+1}(t)|^2 \cdot \sum_j |\bar{y}_j(t)|^2} \le R(t);$$

$$|\sum_j \operatorname{Re}(\bar{y}_j(t) h_j(t,y))| \le \sum_j |y_j(t) h_j(t,y)|$$

$$\le \sqrt{\sum_j |y_j(t)|^2 \sum_j |h_j(t,y)|^2} \le R^{\frac{1}{2}}(t)\,\|h(t,y)\|.$$

Since $\|y(t)\| \le \delta$ and $r(t) < R(t)$ are assumed, we have

$$R^{\frac{1}{2}}(t)\,\|h(t,y)\| \le R^{\frac{1}{2}}(t)\eta\|y(t)\| \le R^{\frac{1}{2}}(t)\eta\sqrt{R(t)+r(t)} \le R^{\frac{1}{2}}(t)\eta\sqrt{2R(t)} \le 2\eta R(t);$$

and

$$\sum_j \mathrm{Re}(\lambda_j y_j(t)\cdot \bar{y}_j(t)) > 6\eta \sum_j y_j(t)\cdot \bar{y}_j(t) = 6\eta \sum_j |y_j(t)|^2 = 6\eta R(t).$$

Therefore, we have the inequality given by

$$\frac{1}{2}R'(t) > 6\eta R(t) - \eta R(t) - 2\eta R(t) = 3\eta R(t).$$

A similar way for $r(t)$ by using $\mathrm{Re}\,\lambda_k \le 0$ yields

$$\frac{1}{2}r'(t) < \eta\, r(t) + 2\eta R(t)$$ (**Detail proof is for homework**).

As long as $\|y(t)\| \le \delta$ and $r(t) < R(t)$, we have

$$\frac{1}{2}(R'(t) - r'(t)) > \eta(3R(t) - r(t) - 2R(t)) = \eta(R(t) - r(t)).$$

Solving this inequality with $r(t_0) < R(t_0)$, we have

$$R(t) - r(t) > (R(t_0) - r(t_0))e^{2\eta(t-t_0)} \quad \text{for all } t \ge t_0 \text{ s.t } \|y(t)\| \le \delta \text{ and } r(t) < R(t).$$

Then

$$\|y(t)\|^2 = \sum_j |y_j(t)|^2 + \sum_k |y_k(t)|^2 \ge \sum_j |y_j(t)|^2 - \sum_k |y_k(t)|^2$$
$$= R(t) - r(t) > (R(t_0) - r(t_0))e^{2\eta(t-t_0)}$$

for all $t \ge t_0$ once $\|y(t)\| \le \delta$ and $r(t) < R(t)$. So this solution $y(t)$ leaves the domain given by $\|y\| \le \delta$, this implies that $x = 0$ is unstable. □

Remark 9.1 The proof of Theorem 9.1 and Theorem 9.2 is by an analytical method. A bit tedious! We may also prove them based on Lyapunov method, which is relatively simple and will be shown later.

3) Linearization

Theorem 9.3 Suppose that $f(x)$ is a function of C^2 and $f(0) = 0$. Then,

1) If all $\mathrm{Re}\,\lambda(A) < 0$, where $A = Df(0)$, $x = 0$ of $x' = f(x)$ is asymptotically

stable;

2) If there exists at least one λ_0 with $\operatorname{Re}\lambda_0(A) > 0$, $x = 0$ of $x' = f(x)$ is unstable.

Proof. It is immediate applications of Theorem 9.1-9.2 when $f(t, x) \equiv f(x)$ for all $t \geq 0$ by linearization. □

Remark 9.2 Linearization results of Theorem 9.3 work for any hyperbolic equilibrium $x = x_0$. $x' = f(x)$ and $x' = Df(x_0)x$ have the same dynamical behavior in the neighborhood of $x = x_0$.

Remark 9.3 When $A = Df(x_0)$ has $\operatorname{Re}\lambda(A) = 0$, the linearization method fails. See the following two examples, which show that anything could be possible when a different perturbation satisfying $\lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0$ is imposed.

Example 9.1 Consider

$$x_1' = -x_2 - x_2(x_1^2 + x_2^2); \quad x_2' = x_1 + x_1(x_1^2 + x_2^2),$$

where $A = Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has $\lambda = \pm i$ with $\operatorname{Re}\lambda(A) = 0$. Since $x = 0$ is a center of $x' = Df(0)x$, it is stable but not asymptotically stable.

Introducing the polar coordinate transformation

$$x_1 = r\cos\theta; \quad x_2 = r\sin\theta,$$

we have ($x_1 \frac{dx_1}{dt} + x_2 \frac{dx_2}{dt} = r\frac{dr}{dt}$; $x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} = r^2 \frac{d\theta}{dt}$; The detail leaves for students)

$$\frac{dr}{dt} = 0; \quad \frac{d\theta}{dt} = 1 + r^2.$$

Solving the equations yields the solution: $r(t) = r_0^2$. So $x = 0$ is still stable ($x = 0$ is a center of the original equations).

Example 9.2 Consider

$$x_1' = x_2 + ax_1(x_1^2 + x_2^2); \quad x_2' = -x_1 + ax_2(x_1^2 + x_2^2),$$

where $a \neq 0$ is a parameter and $A = Df(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has $\lambda = \pm i$ with $\operatorname{Re}\lambda(A) = 0$. It is a center of $x' = Df(0)x$.

Introducing the polar coordinate transformation

$$x_1 = r\cos\theta; \quad x_2 = r\sin\theta,$$

we have

$$\frac{dr}{dt} = ar^3; \quad \frac{d\theta}{dt} = -1.$$

$x = 0$ is a stable focus when $a < 0$ and an unstable focus when $a > 0$.

3. Examples for Stability by Linearization

1) Predator-Prey Model

The Predator-Prey model is given by (Volterra, Italy)

$$x' = x(\alpha - \beta y), \quad y' = y(\gamma x - \delta),$$

where $x > 0$ and $y > 0$; α, β, γ and δ are given positive parameters. $x(t)$ is **the population of the preys** and $y(t)$ is **the population of the predators**.

There are two equilibriums: $(0, 0)$ and $(\frac{\delta}{\gamma}, \frac{\alpha}{\beta})$. Denote $f(x, y) = x(\alpha - \beta y)$, $g(x, y) = y(\gamma x - \delta)$.

For the equilibrium $(0, 0)$, the Jacobian matrix

$$\left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{\substack{x=0 \\ y=0}} = \left.\begin{pmatrix} \alpha - \beta y & -\beta x \\ \gamma y & \gamma x - \delta \end{pmatrix}\right|_{\substack{x=0 \\ y=0}} = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix}$$

has two eigenvalues $\lambda_1 = \alpha$ and $\lambda_2 = -\delta$, which is a saddle point. $(0, 0)$ of the Predator-Prey model is unstable by linearization.

For the equilibrium $(\frac{\delta}{\gamma}, \frac{\alpha}{\beta})$, the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\Bigg|_{\substack{x=\frac{\delta}{\gamma} \\ y=\frac{\alpha}{\beta}}} = \begin{pmatrix} \alpha-\beta y & -\beta x \\ \gamma y & \gamma x-\delta \end{pmatrix}\Bigg|_{\substack{x=\frac{\delta}{\gamma} \\ y=\frac{\alpha}{\beta}}} = \begin{pmatrix} 0 & -\frac{\beta\delta}{\gamma} \\ \frac{\alpha\gamma}{\beta} & 0 \end{pmatrix}$$

has the eigenvalues $\lambda = \pm i\sqrt{\alpha\delta}$ with $\mathrm{Re}\,\lambda(A) = 0$, whose stability can not be determined by linearization.

By separation, we have

$$\frac{\gamma x - \delta}{x} dx = \frac{\alpha - \beta y}{y} dy,$$

Integrating gives trajectories

$$(\gamma x - \delta \ln x) + (\beta y - \alpha \ln y) = \ln C \quad \Leftrightarrow \quad y^{\alpha} e^{-\beta y} = C x^{-\delta} e^{\gamma x}.$$

This can be shown qualitatively that the level curves are bounded and closed. The solutions are periodic.

If $y < \frac{\alpha}{\beta} \Leftrightarrow \alpha - \beta y > 0$, $x(t)$ is increasing by $x' > 0$.

If $y > \frac{\alpha}{\beta} \Leftrightarrow \alpha - \beta y < 0$, $x(t)$ is decreasing by $x' < 0$.

While if $x > \frac{\delta}{\gamma} \Leftrightarrow \gamma x - \delta > 0$, $y(t)$ is increasing by $y' > 0$;

if $x < \frac{\delta}{\gamma} \Leftrightarrow \gamma x - \delta < 0$, $y(t)$ is decreasing by $y' < 0$.

2) Competing Species Model

$$x' = x - ax^2 - cxy, \quad y' = y - by^2 + dxy,$$

where $x \geq 0$ and $y \geq 0$; a, b, c, d are given positive parameters. $x(t)$ is **the population of one species** and $y(t)$ is **the population of the other species**.

If $y = 0$, then $x' = x - ax^2$ is a logistic equation. The population $x(t)$ has a linear growth rate with a natural limit of $x = \frac{1}{a}$. A similar situation holds for $y(t)$ if $x = 0$. The third terms represent interaction between two species.

By solving

$$0 = x - ax^2 - cxy = x(1 - ax - cy)\,; \;\; 0 = y - by^2 + dxy = y(1 - by + dx)\,,$$

We have four equilibriums:

$$(0,0)\,; \;\; (0,\frac{1}{b})\,; \;\; (\frac{1}{a},0)\,; \;\; (\frac{b-c}{ab+cd},\frac{a+d}{ab+cd})\,.$$

If $b \ge c$ (weak interaction) there are four equilibriums in the domain of interest ($x \ge 0$, $y \ge 0$). If $b < c$ (strong interaction) there are only three equilibriums in the domain of interest ($x \ge 0$, $y \ge 0$).

Denote $f(x,y) = x - ax^2 - cxy$ and $g(x,y) = y - by^2 + dxy$.

For the equilibrium $(0,0)$, the Jacobian matrix

$$\left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{\substack{x=0\\y=0}} = \left.\begin{pmatrix} 1-2ax-cy & -cx \\ dy & 1-2by+dx \end{pmatrix}\right|_{\substack{x=0\\y=0}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1$. The origin is a source of the linearized equations. $(0,0)$ is also a source of the competing species model by linearization.

For the equilibrium $(0,\frac{1}{b})$, the Jacobian matrix

$$\left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{\substack{x=0\\y=\frac{1}{b}}} = \left.\begin{pmatrix} 1-2ax-cy & -cx \\ dy & 1-2by+dx \end{pmatrix}\right|_{\substack{x=0\\y=\frac{1}{b}}} = \begin{pmatrix} 1-\frac{c}{b} & 0 \\ \frac{d}{b} & -1 \end{pmatrix}$$

has two eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1 - \frac{c}{b}$. $(0,\frac{1}{b})$ is a saddle point if $b > c$ and a sink if $b < c$. The same dynamical behavior of the competing species model has in the neighborhood of $(0,\frac{1}{b})$ by linearization.

For the equilibrium $(\frac{1}{a},0)$, the Jacobian matrix

$$\left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{\substack{x=\frac{1}{a}\\ y=0}} = \left.\begin{pmatrix} 1-2ax-cy & -cx \\ dy & 1-2by+dx \end{pmatrix}\right|_{\substack{x=\frac{1}{a}\\ y=0}} = \begin{pmatrix} -1 & -\frac{c}{a} \\ 0 & 1+\frac{d}{a} \end{pmatrix}$$

has two eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1 + \frac{d}{a}$. $(\frac{1}{a}, 0)$ is a saddle point no matter of $b > c$ and $b < c$. $(\frac{1}{a}, 0)$ is also a saddle point of the competing species model by linearization.

For the equilibrium $(\frac{b-c}{ab+cd}, \frac{a+d}{ab+cd})$, the Jacobian matrix is

$$\left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{\substack{x=\frac{b-c}{ab+cd}\\ y=\frac{a+d}{ab+cd}}} = \left.\begin{pmatrix} 1-2ax-cy & -cx \\ dy & 1-2by+dx \end{pmatrix}\right|_{\substack{x=\frac{b-c}{ab+cd}\\ y=\frac{a+d}{ab+cd}}}$$

$$= \frac{1}{ab+cd}\begin{pmatrix} -a(b-c) & -c(b-c) \\ d(a+d) & -b(a+d) \end{pmatrix}.$$

For $b > c$ (weak interaction), A has 2 eigenvalues with negative real part because $\lambda_1 \cdot \lambda_2 = \det(A) > 0$ and $\lambda_1 + \lambda_2 = \text{Trance}(A) < 0$. So $(\frac{b-c}{ab+cd}, \frac{a+d}{ab+cd})$ is a stable equilibrium for the competing species model by linearization.

For $b < c$ (strong interaction), since $\lambda_1 \cdot \lambda_2 = \det(A) = (a+d)(b-c) < 0$, then $(\frac{b-c}{ab+cd}, \frac{a+d}{ab+cd})$ is a saddle point of the linearized equations, so does for the competing species model by linearization. So the species $y(t)$ will die out because the equilibrium is a saddle point.

4. Summary

- Linearization works for hyperbolic equilibriums, be effective for local.
- Linearization preserves stability property only, not for trajectory structure.
- The proof method of Theorem 9.1 is typical. Hope to understand and know how.